

Stabilization of ultra-short pulses in cubic nonlinear media

Y. Chung^a and T. Schäfer^b

^a *Department of Mathematics, Southern Methodist University, Dallas, TX 75275, USA*

^b *Department of Mathematics, College of Staten Island,
City University of New York, Staten Island, NY 10314, USA*

(Dated: February 8, 2008)

We study the propagation of ultra-short pulses in a cubic nonlinear medium. Using multiple-scale technique, we derive a new wave equation that preserves the nonlocal dispersion terms present in Maxwell's equations. As a result, we are able to understand how ultra-short nonlinear shocks are stabilized by dispersive terms. A delicate balance between dispersion and nonlinearity leads to a new type of solitary waves. Their stability is confirmed by numerical simulations of full Maxwell's equations.

PACS numbers: 42.81.-i, 42.25.Dd

I. INTRODUCTION

Recent progress in nonlinear optical systems has led to increasing development and use of ultra-short technologies. Driven by the advent of novel detection techniques such as frequency resolved optical gating (FROG), the precise measurement of ultra-short pulses is now possible. Understanding the dynamics of ultra-short pulses has been the subject of intensive research due to their potential uses for various fields, e.g., medical applications and imaging [1], continuum recompression and control from highly nonlinear photonic crystal fibers [2, 3], and the next generation of telecommunication systems.

The cubic nonlinear Schrödinger equation (NLSE) has been the governing model for pulse propagation in nonlinear medium. The NLSE, however, was shown to be inadequate for describing ultra-short pulses. As the pulse power increases, i.e., the pulse becomes shorter, the delay in the response of the media to the excitation of the electric field starts to play a dominant role. The characteristics of the delay are not instantaneous, and hence they need to be described by convolution terms in governing equations. In contrast to differential operators, which are local operators, convolution terms involve nonlocal operators and hence the resulting system is an integro-differential equation. The NLSE is a local equation, namely, it only involves local terms approximating the delayed response. Thus, the NLSE remains valid only while the basic assumption, the existence of fairly broad pulses, is guaranteed [4].

Current optical technology allows to design state-of-the-art optical devices such as photonic crystals or microstructured fibers whose structures are more complex than standard optical fibers. Experimental observations have shown that these new devices provide remarkable phenomena never seen in standard optical fibers. Some of the most prominent phenomena include creation of photonic band gaps, supercontinuum generation with relatively low pulse intensities, ultrashort pulse generation, pulse fission reminiscent of multisoliton breakup [5, 6, 7], and the simultaneous third harmonic generation [8]. While photonic crystal technology can successfully

demonstrate the remarkable performance of the devices, very little is known in theoretical study. It is evident that one must develop a reliable mathematical model that can explain these new optical phenomena. In fact, any envelope approximation that eventually leads to the cubic NLSE fails to exhibit the optical mechanism which evidently goes beyond the local soliton dynamics.

A considerable amount of research efforts have been made to capture the complex optical phenomena that cannot be explained by the cubic NLSE. For short pulse dynamics, for example, one needs to introduce higher order dispersive terms, which lead to modified NLSE whose structure, although much more complex, still remains local. For certain materials, e.g. bulk silica, one can derive a local equation which describes the evolution of ultra-short pulses by making use of the specific form of the material susceptibility, $\chi^{(1)}$ [9]. Recently, the short pulse equation (SPE) was independently derived, which was proven to be a better approximation as the NLSE loses accuracy [10]. The SPE, however, was not shown to possess soliton solutions [9]. Moreover, the SPE is not valid for all materials since the equation is derived taking into account a particular type of the susceptibility of bulk silica. In this paper, considering general forms of susceptibility, we derive a new model without imposing any locality assumption and therefore, avoiding potential loss of smoothness that could have helped to obtain stable solutions in SPE. While soliton theory has been developed in the context of locality, it is now our main task to investigate if nonlocal equations can provide similar phenomena. Indeed, we show that our new nonlocal equation also possesses solitary waves. Furthermore, we perform numerical simulations to confirm the stability of the solutions, which suggests that the soliton theory can be extended to systems with convolution terms.

The paper is organized as follows. In section II, we present the derivation of the nonlinear nonlocal wave equation from Maxwell's equations. In section III, using multiple-scale technique, we derive a weakly nonlinear approximation to the wave equation introduced in section II. In section IV, we show that the absence of dispersion leads to the shock formation. In section V, we

further stabilize the optical shocks by introducing a dispersive convolution term, and show our model possesses solitary waves. In section VI, we confirm the stability of the solitary wave solutions by numerical simulations. Finally, in section VII we summarize the main results of our analysis.

II. DERIVATION OF NONLINEAR NONLOCAL WAVE EQUATION

Prominent examples of effects that lead to convolution terms are optical phenomena due to the retardation in the response of the media to excitation by an oscillating electric field [11]. Recently, the relation between nonlocal terms and quadratic solitons in quadratic nonlinear $\chi^{(2)}$ -materials has been studied [12]. Here, we consider cubic nonlinear $\chi^{(3)}$ materials and investigate their impacts on wave propagation. We start with Maxwell's equations:

$$\begin{aligned}\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, & \nabla \cdot \mathbf{D} &= \rho, \\ \nabla \times \mathbf{H} &= \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t}, & \nabla \cdot \mathbf{B} &= 0.\end{aligned}\quad (2.1)$$

The fields \mathbf{E} , \mathbf{B} , \mathbf{D} , \mathbf{H} are real-valued vectors depending on the three space variables x , y , z and the time t ; ρ is the charge density and \mathbf{j} is the current density vector. We write Φ_1 , Φ_2 , Φ_3 to denote the the first, second, and the third component of the vector Φ . The main results of this paper do not depend on the dimension of the considered system. Thus, for simplicity, the analysis is carried out in an one-dimensional context. This allows us to reduce

the Maxwell's equations to *one* dimensional system of equations by setting

$$\mathbf{E} = E_3(x, t)\mathbf{\tilde{e}}_3, \quad \mathbf{H} = H_2(x, t)\mathbf{\tilde{e}}_2. \quad (2.2)$$

In dielectric media, there exist neither free charges nor field sources, and hence we can set $\mathbf{j} = 0$ and $\rho = 0$. We also assume $\mathbf{B} = \mu_0 \mathbf{H}$ where μ_0 is a constant, which is appropriate for bulk silica. The relation between \mathbf{D} and \mathbf{E} is given by

$$D_3 = \epsilon_0 E_3 + P_3 \quad (2.3)$$

where ϵ_0 is the permeability and P_3 will be defined later. These assumptions reduce the system of equations (2.1) to two one-dimensional equations:

$$\frac{\partial H_2}{\partial x} = \frac{\partial D_3}{\partial t}, \quad \frac{\partial E_3}{\partial x} = \mu_0 \frac{\partial H_2}{\partial t}. \quad (2.4)$$

Using the above relations, we find the equation for E_3 ,

$$\frac{\partial^2 E_3}{\partial x^2} - \epsilon_0 \mu_0 \frac{\partial^2 E_3}{\partial t^2} = \mu_0 \frac{\partial^2 P_3}{\partial t^2}. \quad (2.5)$$

Now we set $c^2 = (\mu_0 \epsilon_0)^{-1}$ and obtain the standard form of the wave equation:

$$\frac{\partial^2 E_3}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 E_3}{\partial t^2} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \left(\frac{P_3}{\epsilon_0} \right). \quad (2.6)$$

Since the response of the medium is not instantaneous, we account for this *retarded response* in terms of convolution integrals and obtain

$$P_3 = \epsilon_0 \int \chi^{(1)}(t - \tau) E_3(x, \tau) d\tau + \epsilon_0 \int \chi^{(3)}(t - \tau_1, t - \tau_2, t - \tau_3) E_3(x, \tau_1) E_3(x, \tau_2) E_3(x, \tau_3) d\tau_1 d\tau_2 d\tau_3. \quad (2.7)$$

Note that this is where convolution terms enter naturally. We finally find one dimensional nonlinear wave equation

(replacing E_3 by E and setting $c = 1$)

$$\frac{\partial^2 E}{\partial x^2} = \frac{\partial^2 E}{\partial t^2} + \frac{\partial^2}{\partial t^2} \left(\int \chi^{(1)}(t - \tau) E(x, \tau) d\tau + \int \chi^{(3)}(t - \tau_1, t - \tau_2, t - \tau_3) E(x, \tau_1) E(x, \tau_2) E(x, \tau_3) d\tau_1 d\tau_2 d\tau_3 \right) \quad (2.8)$$

Here, convolution terms are present in both the dispersive and the nonlinear terms. Generally, the convolution integral in the dispersion has more significant impact on pulse propagation. The Fourier transform helps to convert the convolution present in the dispersive term to a product, however, even if the nonlinearity was local,

it would lead to a convolution of the nonlinear term in Fourier domain.

III. NONLOCAL TERMS IN MULTI-SCALE EXPANSIONS

Multiple-scale technique allows to derive asymptotic expansions for ordinary and partial differential equations [13]. The basic idea is to use the slow scales to remove resonances which are responsible for short validity range of the chosen expansion. In this paper, we focus on the case where dispersion and nonlinearity effects are weak and of the same order ϵ . The method presented here is very general and hence, can be applied to other cases. In

the presence of weak dispersion and nonlinearity, the calculations are more straightforward to perform since the zero-order operator is the unperturbed linear wave equation whose kernel can be constructed by simple exponential functions. This makes it easy to apply the Fredholm alternative leading to the solvability conditions which result in the equations for the slow evolution of the system.

Let us write the basic wave equation again, this time with a small dispersive and small nonlinear term on the right hand side:

$$\frac{\partial^2 E}{\partial x^2} = \frac{\partial^2 E}{\partial t^2} + \epsilon \frac{\partial^2}{\partial t^2} \left(\int \chi^{(1)}(t - \tau) E(x, \tau) d\tau + \int \chi^{(3)}(t - \tau_1, t - \tau_2, t - \tau_3) E(x, \tau_1) E(x, \tau_2) E(x, \tau_3) d\tau_1 d\tau_2 d\tau_3 \right) \quad (3.1)$$

We introduce multiple scales *only* in the evolution variable z , and thus no additional scales in t . Then, in contrast to the derivation of the NLSE, our analysis is valid even in the case where the Fourier transform is not concentrated around one special frequency. This is of particular importance for the study of very short pulses or phenomena involving frequency-mixing. The first step is to rewrite (3.1) in Fourier domain

$$\left(\frac{\partial^2}{\partial x^2} + \omega^2 \right) \hat{E} = -\epsilon \omega^2 \hat{\chi}^{(1)}(\omega) \hat{E}(x, \omega) - \epsilon \frac{\omega^2}{(2\pi)^2} \mathcal{N}(E), \quad (3.2)$$

where we have introduced the nonlinear (and nonlocal) operator

$$\mathcal{N}(E) = \int \hat{\chi}^{(3)}(\omega_1, \omega_2, \omega_3) \hat{E}(x, \omega_1) \hat{E}(x, \omega_2) \hat{E}(x, \omega_3) \times \delta(\omega - \omega_1 - \omega_2 - \omega_3) d\omega_1 d\omega_2 d\omega_3. \quad (3.3)$$

Now we perform a multiple scale expansion

$$\begin{aligned} \hat{E}(x, \omega) &= \hat{E}_0(x_0, x_1, \omega) + \epsilon \hat{E}_1(x_0, x_1, \omega) + \dots, \\ x_0 &= x, \quad x_1 = \epsilon x, \end{aligned} \quad (3.4)$$

and solve Eq. (3.2) order by order in ϵ . The zero-order in ϵ yields

$$\left(\frac{\partial^2}{\partial x_0^2} + \omega^2 \right) \hat{E}_0 = 0 \quad (3.5)$$

with the solution

$$\hat{E}_0(x_0, x_1, \omega) = \hat{A}_0(x_1, \omega) e^{i\omega x} + \hat{B}_0(x_1, \omega) e^{-i\omega x} \quad (3.6)$$

where the term containing \hat{A}_0 corresponds to a wave moving to the right and \hat{B}_0 to a wave moving to the left.

Collecting the first order terms in ϵ , we find the equation for \hat{E}_1 as

$$\begin{aligned} \left(\frac{\partial^2}{\partial x_0^2} + \omega^2 \right) \hat{E}_1 &= -2 \frac{\partial}{\partial x_0} \frac{\partial}{\partial x_1} \hat{E}_0 - \omega^2 \hat{\chi}^{(1)}(\omega) \hat{E}_0 \\ &\quad - \frac{\omega^2}{(2\pi)^2} \mathcal{N}(E_0). \end{aligned} \quad (3.7)$$

The question is now how to formulate the solvability condition for (3.7) taking into account the nonlocal driving term on the right hand side of the equation. The fundamental solutions of the linear operator on the l.h.s. of Eq. (3.7) are again $\{\exp(i\omega x_0), \exp(-i\omega x_0)\}$. Following the technique of multiple scales, we apply Fredholm alternative for removing possible resonances which arise from the terms in r.h.s. of Eq. (3.7) proportional to those fundamental solutions. Using the expression (3.6), the triple product (omitting the arguments x_0 and x_1 for simplicity) $E_0(\omega_1) E_0(\omega_2) E_0(\omega - \omega_1 - \omega_2)$ will yield eight terms. In each of those terms, we find all the possible combinations of ω_1 and ω_2 that lead to the solution \hat{E}_1 proportional to either $\exp(i\omega x_0)$ or $\exp(-i\omega x_0)$. Those combinations are responsible for resonance and hence, must be removed. From Eq. (3.6), we find (the primes denote derivatives with respect to the slow variable x_1)

$$-2 \frac{\partial}{\partial x_0} \frac{\partial}{\partial x_1} \hat{E}_0(x_0, x_1, \omega) = -2i \left(\hat{A}'_0 e^{i\omega x_0} + \hat{B}'_0 e^{-i\omega x_0} \right). \quad (3.8)$$

Thus, the contributions proportional to $\exp(i\omega x_0)$, $\exp(-i\omega x_0)$ will appear in the equation for \hat{A}'_0 and \hat{B}'_0 , respectively. Let us now observe the first of the eight terms (omitting the evolution variable x_0 for the sake of easy typesetting)

$$\hat{A}_0(\omega_1) \hat{A}_0(\omega_2) \hat{A}_0(\omega - \omega_1 - \omega_2) e^{i\omega x_0}.$$

The oscillations arise from $e^{i\omega x_0}$ and thus, this term will appear *only* in the equation for \hat{A}'_0 . The next term

$$\hat{A}_0(\omega_1) \hat{A}_0(\omega_2) \hat{B}_0(\omega - \omega_1 - \omega_2) e^{i(-\omega + 2\omega_1 + 2\omega_2)x_0}$$

incorporates oscillations with different frequencies corresponding to $e^{i(-\omega + 2\omega_1 + 2\omega_2)x_0}$. Contributions to resonance terms occur if

1. $-\omega + 2\omega_1 + 2\omega_2 = \omega$ responsible for contributions to the equation for \hat{A}'_0 .

2. $-\omega + 2\omega_1 + 2\omega_2 = -\omega$ responsible for contributions to the equation for \hat{B}'_0 .

Therefore, the solvability conditions lead to the system of equations for \hat{A}'_0 and \hat{B}'_0 ,

$$\begin{aligned} -2i\omega\hat{A}'_0(\omega) &= \omega^2\hat{\chi}^{(1)}\hat{A}_0(\omega) + \frac{\omega^2}{(2\pi)^2}\mathcal{N}(A_0) + \\ &\frac{\omega^2}{(2\pi)^2}\left(3\hat{B}_0(0)\int\hat{\chi}^{(3)}(\omega_1,\omega-\omega_2,0)\hat{A}_0(\omega_1)\hat{A}_0(\omega-\omega_1)d\omega_1\right. \\ &\left.+3\hat{A}_0(\omega)\int\hat{\chi}^{(3)}(\omega_1,-\omega_1,\omega)\hat{B}_0(\omega_1)\hat{B}_0(-\omega_1)d\omega_1\right), \end{aligned} \quad (3.9)$$

$$\begin{aligned} 2i\omega\hat{B}'_0(\omega) &= \omega^2\hat{\chi}^{(1)}(\omega)\hat{B}_0(\omega) + \frac{\omega^2}{(2\pi)^2}\mathcal{N}(B_0) + \\ &\frac{\omega^2}{(2\pi)^2}\left(3\hat{A}_0(0)\int\hat{\chi}^{(3)}(\omega_1,\omega-\omega_2,0)\hat{B}_0(\omega_1)\hat{B}_0(\omega-\omega_1)d\omega_1\right. \\ &\left.+3\hat{B}_0(\omega)\int\hat{\chi}^{(3)}(\omega_1,-\omega_1,\omega)\hat{A}_0(\omega_1)\hat{A}_0(-\omega_1)d\omega_1\right). \end{aligned} \quad (3.10)$$

The resulting system of equations for \hat{A}_0 and \hat{B}_0 describes the slow evolution of Eq. (2.8). These provide a significant simplification of the original equation (2.8), which are the *first* order equations in x_1 and, if we take the inverse Fourier transform, first order in t as well. Note that the resulting equations preserve the nonlocal structure, which was our main purpose. This pair of equations can be seen as an analog to the NLSE valid for broad pulses (without the assumption of a signal centered around a specific carrier frequency) or the SPE valid for short pulses (without the specific assumption about the profile of linear and nonlinear susceptibility). The only assumption necessary to obtain the above system was the smallness of the dispersive and the nonlinear terms. Our derivation also suggests how to proceed in the case where the dispersive term is not small. In this case, the solutions of the linear problem will involve a more complicated dispersion relation which will give rise to modified resonance conditions for the nonlinearity.

Note that Eqs. (3.9), (3.10) also account for coupling between the forward and backward moving waves A_0 , B_0 , respectively. The coupling is weak in the sense that it only consists of integrated quantities. In some cases, however, e.g. when the susceptibilities have an additional dependence on the evolution variable x such as in certain photonic crystal structures, the coupling between A_0 and B_0 can become important.

IV. THE DISPERSION-FREE CASE

In the search of soliton solutions of the underlying non-local equations, we first consider a particular case, absence of linear dispersion $\hat{\chi}^{(1)} = 0$ and constant nonlinear susceptibility $\hat{\chi}^{(3)} = 1$. Then, we take the inverse Fourier

transform of (3.9), (3.10) to obtain differential equations in time domain.

We first notice that $\hat{B}'_0(0, x_1) = 0$ implies $\hat{B}_0(0) = \hat{B}_0(\omega = 0, x_1)$ is constant as a function of x_1 and the corresponding statement also holds for $\hat{A}_0(\omega = 0, x_1)$. For convenience of notation, let us define

$$2\pi B_{\text{zero}} = \hat{B}_0(0), \quad 2\pi A_{\text{zero}} = \hat{A}_0(0).$$

After straightforward calculation we find that

$$\begin{aligned} A_{\text{int}} &:= \frac{1}{(2\pi)^2} \int \hat{A}_0(\omega_1)\hat{A}_0(-\omega_1) d\omega_1, \\ B_{\text{int}} &:= \frac{1}{(2\pi)^2} \int \hat{B}_0(\omega_1)\hat{B}_0(-\omega_1) d\omega_1 \end{aligned} \quad (4.1)$$

are constants of motion as well. Therefore, after Fourier transform back in time domain we have ($A_0 = A_0(x, t)$ and $B_0 = B_0(x, t)$)

$$\frac{\partial}{\partial x_1} A_0 + \frac{1}{2} \frac{\partial}{\partial t} (A_0^3 + 3B_{\text{zero}}A_0^2 + 3B_{\text{int}}A_0) = 0, \quad (4.2)$$

$$\frac{\partial}{\partial x_1} B_0 - \frac{1}{2} \frac{\partial}{\partial t} (B_0^3 + 3A_{\text{zero}}B_0^2 + 3A_{\text{int}}B_0) = 0. \quad (4.3)$$

As mentioned before, the forward and backward propagating waves A_0 , B_0 are coupled. This coupling, however, is weak in a sense that it occurs only through the constants of motions. By choosing initial condition $B_0(x_1 = 0, t) \equiv 0$ we can eliminate the backward propagating wave B_0 entirely at the leading order and the equation for A_0 is reduced to a Burger's type equation

$$\frac{\partial}{\partial x_1} A_0 + \frac{1}{2} \frac{\partial}{\partial t} A_0^3 = 0, \quad (4.4)$$

with the (implicit) solution

$$A_0(t, x_1) = f\left(t - \frac{3}{2}A_0^2x_1\right). \quad (4.5)$$

Here, the function f is determined by the initial condition of the pulse at $t = 0$. For typical pulse profiles (e.g., a Gaussian pulse), we see that shocks will be generated eventually. This was expected as we excluded all the linear dispersion terms from our considerations that could have helped to smoothen the solution.

V. SOLITARY WAVES

In order to regularize the shocks and further obtain stable wave propagation, we bring back a small dispersion term ($\hat{\chi}^{(3)}$ is still set to be constant for simplicity) and find

$$\begin{aligned} \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right) E(x, t) &= \epsilon \frac{\partial^2}{\partial t^2} \int \chi^{(1)}(t - \tau) E(x, \tau) d\tau \\ &+ \epsilon \frac{\partial^2}{\partial t^2} E(x, t)^3. \end{aligned} \quad (5.1)$$

Now we apply the same analysis used before and obtain an additional nonlocal term in Eqs. (4.2), (4.3),

$$\frac{\partial}{\partial x_1} A_0 + \frac{1}{2} \frac{\partial}{\partial t} (A_0^3 + 3B_{\text{zero}} A_0^2 + 3B_{\text{int}} A_0 + \int \chi^{(1)}(t - \tau) A_0(x_1, \tau) d\tau) = 0, \quad (5.2)$$

$$\frac{\partial}{\partial x_1} B_0 - \frac{1}{2} \frac{\partial}{\partial t} (B_0^3 + 3A_{\text{zero}} B_0^2 + 3A_{\text{int}} B_0 + \int \chi^{(1)}(t - \tau) B_0(x_1, \tau) d\tau) = 0. \quad (5.3)$$

Again we assume the back-scattered part B_0 is suppressed. Motivated by the recent developments e.g., in photonic crystal structures a wide range of susceptibilities can be engineered, we search for the profiles of A_0 and $\chi^{(1)}$ which can stabilize the shock formation. One simple approach to finding a stationary solution is to set $\partial A_0 / \partial x_1 = 0$. This implies we need to find A_0 and $\chi^{(1)}$ such that

$$A_0^3 + \int \chi^{(1)}(t - \tau) A_0(x_1, \tau) d\tau = 0. \quad (5.4)$$

Denoting $\hat{\chi}^{(1)} = \mathcal{F}(\chi^{(1)})$, we rewrite Eq. (5.4) in Fourier domain,

$$\mathcal{F}(A_0^3) + \hat{\chi}^{(1)} \hat{A}_0 = 0. \quad (5.5)$$

If we assume a Lorentz profile for A_0

$$A_0(t) = \frac{\alpha}{1 + (\beta t)^2}, \quad (5.6)$$

we can solve Eq. (5.4) using Fourier transform. The Fourier transform of A_0 is

$$\hat{A}_0(\omega) = \pi \frac{\alpha}{\beta} e^{-|\omega|/\beta} \quad (5.7)$$

and, using the residue theorem, we obtain the Fourier transform of A_0^3 ,

$$\mathcal{F}(A_0^3)(\omega) = \frac{\pi}{8} \frac{\alpha^3}{\beta^3} e^{-|\omega|/\beta} (\omega^2 + 3\beta|\omega| + 3\beta^2). \quad (5.8)$$

We find that Eq. (5.6) is a stationary solution provided that the susceptibility takes the following parabolic form:

$$\hat{\chi}^{(1)}(\omega) = -\frac{1}{8} \frac{\alpha^2}{\beta^2} (\omega^2 + 3\beta|\omega| + 3\beta^2). \quad (5.9)$$

We expect that a pulse of this particular shape, for the given susceptibility will not undergo changes at the leading order, hence propagates as a stable nonlinear wave. The dispersion compensates for the effect of nonlocal nonlinear term not only in a small frequency range but for all possible values of ω . Due to this balance between convolution terms in the nonlinearity and dispersion, we can obtain a stable solitary wave.

Most of the physical examples for Maxwell's equations, however, require a modulated initial condition. Therefore, we consider a pulse of the form

$$\tilde{A}_0(t) = A_0(t) \cos(\omega_0 t) \quad (5.10)$$

which will have its maximal frequency coefficients in the neighborhood of ω_0 . We extend our analysis to this case of a modulated initial condition and find a susceptibility for which the given pulse will maintain its stable propagation. Since the Fourier transforms of \tilde{A}_0 and \tilde{A}_0^3 can be found directly as

$$\mathcal{F}(\tilde{A}_0) = \frac{1}{2} (\hat{A}_0(\omega - \omega_0) + \hat{A}_0(\omega + \omega_0)) \quad (5.11)$$

$$\begin{aligned} \mathcal{F}(\tilde{A}_0^3) &= \frac{1}{8} (\mathcal{F}(A_0^3)(\omega + 3\omega_0) + 3\mathcal{F}(A_0^3)(\omega + \omega_0) \\ &\quad + 3\mathcal{F}(A_0^3)(\omega - \omega_0) + \mathcal{F}(A_0^3)(\omega - 3\omega_0)) \end{aligned} \quad (5.12)$$

we find the susceptibility $\mathcal{F}(\tilde{\chi}^{(1)})$ for the modulated solitary wave as

$$\mathcal{F}(\tilde{\chi}^{(1)}) = -\frac{\mathcal{F}(\tilde{A}_0^3)}{\mathcal{F}(\tilde{A}_0)} \quad (5.13)$$

It is important to note that the presented susceptibilities (5.9) and (5.13) do not correspond to susceptibilities of any known material. Although they need to be engineered artificially, recent progress in manufacturing particular photonic crystal structures [14, 15, 16] makes it reasonable to expect the possibility to design such and similar susceptibilities in a not too distant future. On the other hand, the above analysis offers a variety of different approaches for regularizing shocks to obtain stable propagation of nonlinear waves. One possibility, for example, would be to introduce a dependence of the slow variable x_1 to $\chi^{(1)}$ (as in certain photonic crystal structures). This increases the mathematical difficulty to find solutions to the system of equations (5.2), (5.3) but leads to more freedom in the choice of A_0 and $\chi^{(1)}$.

VI. NUMERICAL RESULTS

In order to show the stability of the solitary wave (5.6) numerically, we use its analytic expression together with the corresponding susceptibility (5.9) for simulations of the nonlocal nonlinear wave equation (2.8). Here, we chose $\alpha = 0.2$ and $\beta = 0.75$ leading to an effective ϵ of around 0.07. Transmitting the solitary wave to a distance $x = 25$, which is longer than $\mathcal{O}(1/\epsilon)$, enables us to capture all three important effects: Pulse distortion due to dispersion, pulse distortion due to nonlinearity, and stable propagation of the nonlinear soliton due to a nonlocal balance of dispersive and nonlinear effects. Figure 1 presents the pure linear and nonlinear effects. In Fig. 2, we present the propagation of the solitary wave combining the linear and nonlinear effects. Although the characteristics of the soliton are resulted from a first-order

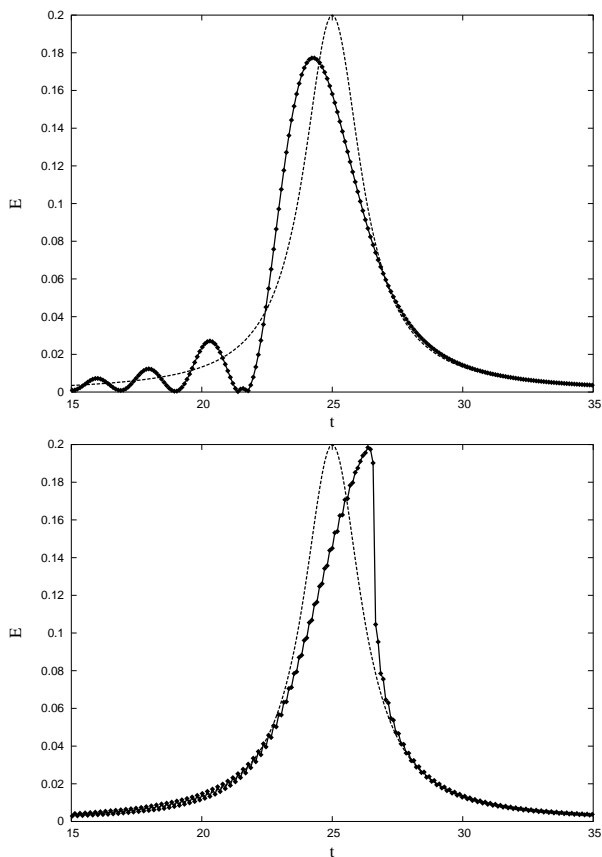


FIG. 1: Figure on top: The diamonds show the solution of Maxwell's wave equation (2.8) in the linear case $\chi^{(3)} = 0$ versus the initial pulse (dashes) in a moving frame. The linear susceptibility $\chi^{(1)}$ leads to dispersion. Bottom figure: The diamonds represents the solution of Maxwell's wave equation (2.8) in the purely nonlinear case $\chi^{(1)} = 0$ versus the initial pulse (dashes) in a moving frame. It clearly illustrates the formation of a shock.

approximation, the numerical simulations show that the balance of nonlinear convolution terms and dispersion generates a stable solution whose shape changes only slightly during the propagation.

The second numerical simulation shows the stable propagation of the modulated initial pulse given by (5.10) and corresponding susceptibility (5.13). Here, we chose simulation parameters $\alpha = 0.2$ and $\beta = 1.0$ and $\omega_0 = 5.0$. The allowed frequency range was restricted to frequencies below $2\omega_0$ to suppress generation of third harmonics. Figure 3 shows how well nonlinearity and dispersion are balanced in this case. The pulse experiences only very small deviations from its original shape that are due to higher-order effects. From this figure it is also clear that this solitary wave represents an ultrashort pulse, hence does not fall into the regime of the cubic nonlinear Schrödinger equation. It is a new type of ultra short solitary wave that can exist due to the balance of dispersion and nonlinearity in a very broad frequency range.

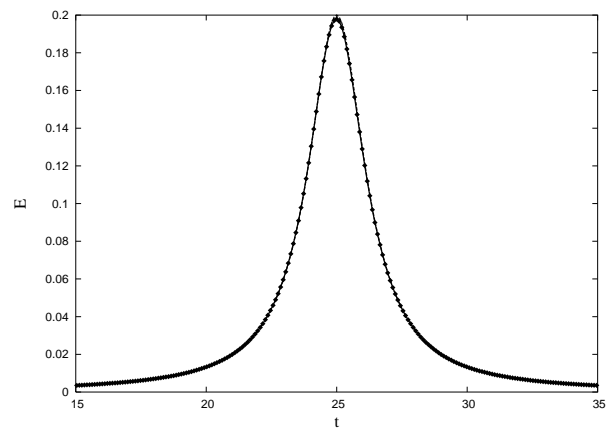


FIG. 2: Propagation of the solitary wave. The diamonds represent the solution of Maxwell's wave equation with both dispersive and nonlinear terms. The stability of the pulse propagation is due to the balance of both effects.

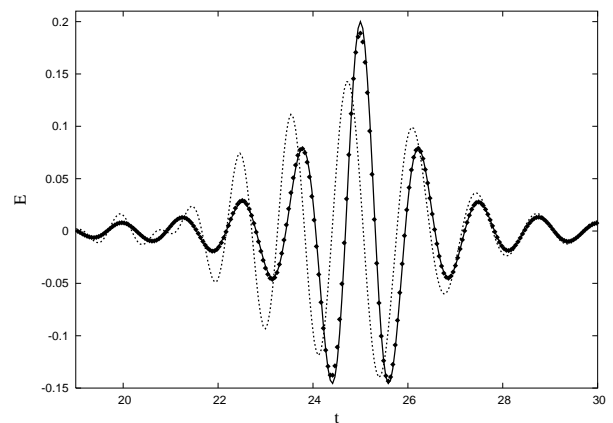


FIG. 3: Propagation of the modulated solitary wave. The diamonds represent the solution of Maxwell's wave equation with both dispersive and nonlinear terms, the continuous line the initial condition in a moving frame. The dashed line is the linear solution of Maxwell's equations with the same initial condition. The figure shows how nonlinearity allows the solitary wave to maintain its initial shape.

VII. CONCLUSION

We derived a slowly varying envelope equation for ultra-short pulses from the cubic nonlinear Maxwell's equation. The derivation was carried out without imposing any locality assumption unlike NLSE, which allows to preserve the phenomena that are due to the retarded response of the media in ultra-short pulse dynamics. We also did not assume any specific types of material susceptibility and hence, our new model can be applied more generally than the SPE. We showed that for certain susceptibility of material the resulting approximative equations possess a solitary wave solution due to a (nonlocal) balance between dispersion and nonlinearity. The stable propagation of this solitary wave is confirmed by numer-

ical simulations based on the full Maxwell's equation.

Acknowledgments

We are grateful to A. Aceves for valuable comments and useful discussions. The work of T. Schäfer was sup-

ported by CUNY research foundation through the grant PSCOC-36-176. The work of Y. Chung was supported by National Science Foundation through the grant NSF-DMS-0505618.

-
- [1] E. B. Brown, E. Wu, W. Zipfel, and W. W. Webb. Measurement of molecular diffusion in solution by multiphoton fluorescence photobleaching recovery. *Biophys. Jour.*, 77:2837–2849, 1999.
 - [2] M. W. Kimmel, R. Trebino, J. Ranka, and A. J. Stentz. *CLEO 2000, CFL7, San Francisco*.
 - [3] J. C. Knight, J. Broeng, T. A. Birks, and P. St. J. Russell. Photonic band gap guidance in optical fibers. *Science*, 282:1476–1478, 1998.
 - [4] J. E. Rothenberg. Space-time focusing: breakdown of the slowly varying envelope approximation in the self-focusing of femtosecond pulses. *Opt. Lett.*, 17:1340–1342, 1992.
 - [5] A. V. Husakou and J. Herrmann. Supercontinuum generation of higher-order solitons by fission in photonic crystal fibers. *Phys. Rev. Lett.*, 87:203901, 2001.
 - [6] J. Herrmann, U. Griebner, N. Zhavoronkov, A. Husakou, D. Nickel, J. C. Knight, and W. J. Wadsworth. Experimental evidence for supercontinuum generation by fission of higher-order solitons in photonic crystal fibers. *Phys. Rev. Lett.*, 88:173901, 2002.
 - [7] J. M. Dudley, X. Gu, L. Xu, M. Kimmel, E. Zeek, P. O'Shea, R. Trebino and S. Coen, and R. S. Windeler. Cross-correlation frequency resolved optical gating analysis of broadband continuum generation in photonic crystal fiber: simulations and experiments. *Opt. Exp.*, 10:1215–1221, 2002.
 - [8] F. G. Omenetto, A. J. Taylor, M. D. Moores, J. Arriaga, J. C. Knight, W. J. Wadsworth, and P. St. Russell. Simultaneous generation of spectrally distinct third harmonics in a photonic crystal fiber. *Opt. Lett.*, 26:1158–1160, 2001.
 - [9] T. Schäfer and C. E. Wayne. Propagation of ultra-short optical pulses in cubic nonlinear media. *Physica D*, 196:90–105, 2004.
 - [10] Y. Chung, C.K.R.T. Jones, T. Schäfer, and C. E. Wayne. Ultra-short pulses in linear and nonlinear media. *Nonlinearity*, 18:1351–1374, 2005.
 - [11] R. W. Boyd. *Nonlinear Optics*. Academic Press, Boston, 1992.
 - [12] N. I. Nikolov, D. Neshev, O. Bang, and W. Z. Krolikowski. Quadratic solitons as nonlocal solitons. *Phys Rev E*, 68:036614, 2003.
 - [13] M. H. Holmes. *Introduction to Perturbation Methods*. Springer, New York, 1995.
 - [14] S. Mingaleev and Y. Kivshar. Nonlinear photonic crystals; toward all-optical technologies. *Optics & Photonics News*, 13, July 2002.
 - [15] A. Bjarklev, J. Broeng, and A. S. Bjarklev. *Photonic Cryatal Fibers*. Kluwer Academic Publishers, Norwell, Boston, 2003.
 - [16] K. P. Hansen, J. Broeng, P. M. W. Skovgaard, J. R. Folkenberg, M. D. Nielsen, A. Peterson, T. P. Hansen, C. Jakobsen, H. R. Simonsen, J. Limpert, and F. Salin. High-power photonic crystal fiber lasers: Design, handling and subassemblies. *Photonics West*, San Jose, CA, 2005.